Mathematics: analysis and approaches
Higher level
Paper 3

Name

Date: ____________________

1 hour

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

exam: 3 pages

Answer all questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 29]

Six balls numbered 1, 2, 2, 5, 5, 7 are put into a bag. For each part of this question, balls are randomly selected from the bag one at a time ***with*** replacement. That is, after a ball is selected from the bag the number on it is noted and it is put back into the bag before another selection is made.

(a) A single ball is taken out of the bag. Let X represent the number shown on the ball. Find the value of $\mathrm{E}(X)$. [2]

(b) Three balls are selected from the bag **with replacement**.

(i) Show that there are three different ways that the sum of the three numbers can be 5. [3]

(ii) Find the probability that the sum of the three numbers is 5. [3]

(iii) Show that the probability that the sum of the three numbers is 9 is $\frac{1}{8}$. [3]

(iv) Find the probability that the median of the three numbers is 1. [3]

(c) Ten balls are selected from the bag **with replacement**. Find the probability that less than four of the selections display the number 5. [3]

(d) (i) n balls are selected **with replacement**. The probability that at least one selection displays the number 5 is greater than 0.95. Show that the minimum value of n is 8. [2]

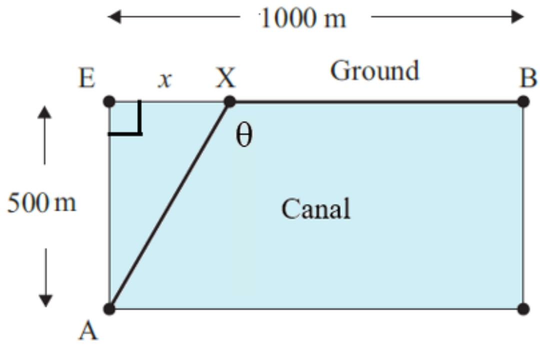
(ii) Find the least number of balls that must be selected from the bag for the probability of at least one ball displaying the number 7 to be greater than 0.85. [3]

(e) Another bag contains k balls numbered 1, 2, 5 or 7. Eight balls are randomly selected from this bag **with replacement**. It is calculated that when selecting eight balls the expected number of balls displaying the number 1 is exactly 4.8, and the variance of the number of balls displaying the number 2 is exactly 1.5. When each of the eight selections occur the probability of selecting a ball that is numbered 1 or 2 is equal to W. Find the value of W. [7]

2. [Maximum mark: 26]

An engineering company is constructing a pipeline to connect two fuel stations (station A and station B) that are on opposite parallel sides of a canal that is 500 meters wide. The pipeline is to be built in two straight sections: one section under the canal from A to X, and another section under the ground from X to B. The cost per meter for the pipeline under the canal is six times the cost per meter for the pipeline under the ground.

As shown in the diagram below, there is a point E on the same side of the canal as station B that is directly across the canal from station A; so that angle AEB is a right angle. The distance from point E to station B is 1000 meters. Let $\text{EX} = x$.



Let p be the cost, in dollars per meter, of constructing the pipeline under the ground from X to B.

(a) Show that the total cost C, in dollars, of building the entire pipeline (AX + XB) is given by

$$C = 6p\sqrt{x^2 + 250000} + 1000p - px$$ [2]

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence, find the value of x such that the total cost is a minimum. Justify that for this value of x the total cost C is a minimum. [7]

(c) Find the minimum total cost in terms of p. [1]

The two straight sections of pipe are joined at X. Let angle $\text{AXB} = \theta$.

(d) Find θ, in degrees, for the value of x calculated in part (b). [2]

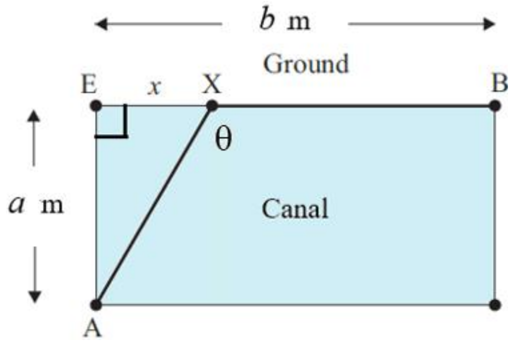
Engineers determine that θ must be at least $120°$ for the pipeline to work safely.

(e) Given this new constraint, find:

(i) the new value of x such that the total cost is a minimum.

(ii) the percentage increase in the minimum total cost compared to the result found in (c). [4]

Consider a more general case where the width of the canal is a meters and $\text{EB} = b$ meters, as shown in the diagram. Again, the cost per meter for the pipeline under the canal is six times the cost per meter for the pipeline under the ground.



(f) With no constraint on angle θ, find an expression for the value of x such that the total cost is a minimum. [6]

(g) Using your result from part (f), find θ in degrees. [2]

(h) Comment on how the parameters a and b affect the value of x such that the total cost is a minimum. [2]